Some geometry of irregular connections on curves

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Guiding principle for seeking hyperkahler manifolds – (Hitchin, Seminar Bourbaki '91):-

"Look for hyperkahler structures on manifolds with complex symplectic structures"

Representative collection of examples:-

1. Resolutions of rational double point singularities
2. Coadjoint orbits of complex Lie groups
3. Spaces of representations of a surface group in a complex Lie group

⋮

- Here a 'surface group' is the fundamental group $\pi_1(\Sigma)$ of a compact (Riemann) surface Σ
- If G a complex (reductive) group

$$M = \mathrm{Hom}^{irr}(\pi_1(\Sigma), G)/G$$

has a natural complex symplectic structure
(Atiyah-Bott, Goldman, ...)
– independent of complex structure of Σ
"symplectic nature of the fundamental group"

Tannakian categories

- Rough idea: axiomatise properties of categories of representations of groups
- Prove:

  any Tannakian category $\cong$ category of representations of some group

Basically: $\pi_1(\Sigma)$ is the Tannaka group of the (Tannakian) category of vector bundles with holom. connections $/\Sigma$

Define 'wild fundamental group' $w\pi_1(\Sigma)$ to be Tannaka group of category of vector bundles with mero. connections $/\Sigma$

– constructed (in some sense) by Martinet-Ramis '91

Picture

$$\mathrm{Hom}(w\pi_1(\Sigma), G)/G$$

much too big

- Has natural Poisson structure & finite dimensional symplectic leaves
- Describe (& study) generic such leaves directly

Main results

① Atiyah-Bott type definition of symplectic str.
- indept. of cx str. of Σ (& "irreg. types")

"symplectic nature of wild fundamental group"

② (w. O. Biquard) HyperKahler metrics & correspondence w. mero. Higgs bundles

(one direction done earlier by Sabbah)

(& simple poles by Simpson, Nakajima, ...)

Alternative viewpoint

"Hyperkahler metrics on total spaces of algebraically completely integrable systems"

- The Hitchin integrable system on moduli spaces of Higgs bundles was extended to meromorphic Higgs bundles by Bottacin, Markman ~ '94

($\mathbb{P}^1$ case by Beauville, Adams-Harnad-Hurtubise, ...)

Plan for rest of talk

1. Simplest examples

2. Quick general description of the spaces

3. More examples

   - surprising link to Poisson Lie groups
   - appearance of G-braid groups

4. Fourier-Laplace transform (simplest case)

5. Generalised Deligne-Simpson problem

Examples

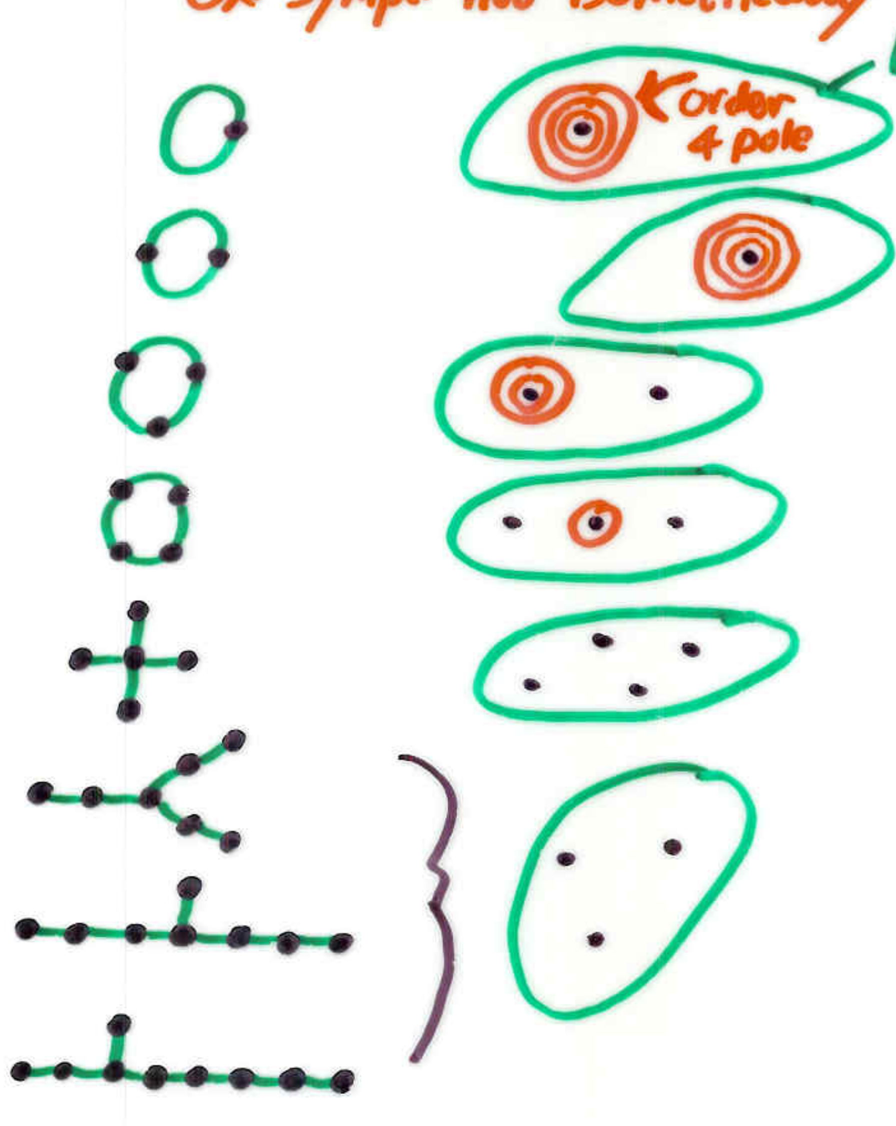
"Quaternionic Curves"

$\left\{ \begin{array}{l} \dim_{\mathbb{H}} = 1 \\ \text{real 4 mfds} \end{array} \right.$

- simplest examples have open pieces

$\cong$ Kronheimer's ALE spaces

$\{$ cx. sympl. not isometrically



A_0	GL_2 - not generic
A_1	GL_2
A_2	GL_2
A_3	GL_2
D_4	GL_2
E_6	GL_3
E_7	GL_4 } not generic
E_8	GL_6 } not generic

cf. Cherkis-Kapustin, ..., Painlevé...

Quick explicit description of the spaces

Σ genus g, m punctures $\sim$ (simple pole case)

Choose m conjugacy classes $\mathcal{C}_i \subset G$

$$\mathcal{M} = \mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)/G$$

$$\cong \left\{ \underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C} \;\middle|\; \underbrace{\prod [A_i, B_i] \prod C_j = 1}_{\text{view as multiplicative moment map condition}},\ C_j \in \mathcal{C}_j \right\} \Big/ G$$

$$=: \underbrace{\mathbb{D} \circledast \cdots \circledast \mathbb{D}}_{g} \circledast \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m /\!\!/ G$$

where $\mathbb{D} = G \times G$ with map $\mu : \mathbb{D} \to G;\ (A,B) \mapsto ABA^{-1}B^{-1}$

– can construct cx sympl. structure algebraically this way

(quasi-Hamiltonian geometry – Alekseev-Malkin-Meinrenken)

Generic higher order poles :-

(most polar coeff. regular semisimple $\in \mathfrak{t}_{reg}$)

Replace conjugacy classes $\mathcal{C} \subset G$ by

$$\mathcal{C} = \left((U_+ \times U_-)^{k-1} \times G \right) / H$$

where $k =$ pole order

$H \subset G$ max. complex torus

$U_\pm \subset B_\pm \subset G$ full unipotent subgroups $(B_+ \cap B_- = H)$

Choose $t \in H$, then 'moment map' is

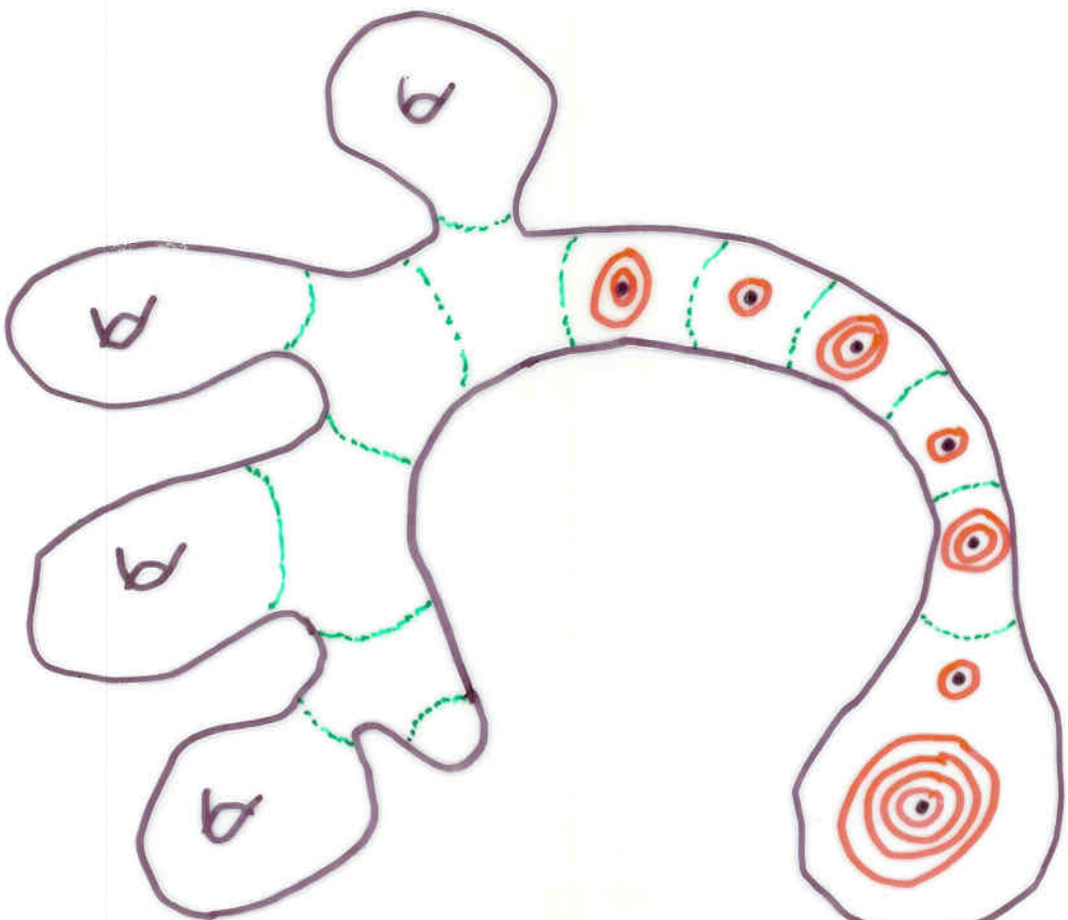
$$\mu : \mathcal{C} \to G ; \quad [\{S_i^\pm\}, C] \mapsto C S_1^- S_1^+ S_2^- S_2^+ \cdots t C^{-1}$$

($\mathcal{C}$ is a quasi-Hamiltonian G-space)

Have such 'generalised conjugacy class' $\mathcal{C}$ at each pole, then

$$M \cong \underbrace{\mathbb{D} \circledast \cdots \circledast \mathbb{D}}_{g} \circledast \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m /\!/ G$$

- get explicit (algebraic) symplectic structure
- same as that from extended Atiyah-Bott viewpoint



Local irregular Riemann-Hilbert correspondence

(generic, order 2 pole)

- $\Delta \subset \mathbb{C}$ unit disk
- $\mathcal{G} = \{$ holomorphic maps $g : \Delta \to G\}$
- $\mathcal{G}_1 = \{ g \in \mathcal{G} \mid g(0) = 1 \}$
- Fix $A_0 \in \mathfrak{h}_{reg}$

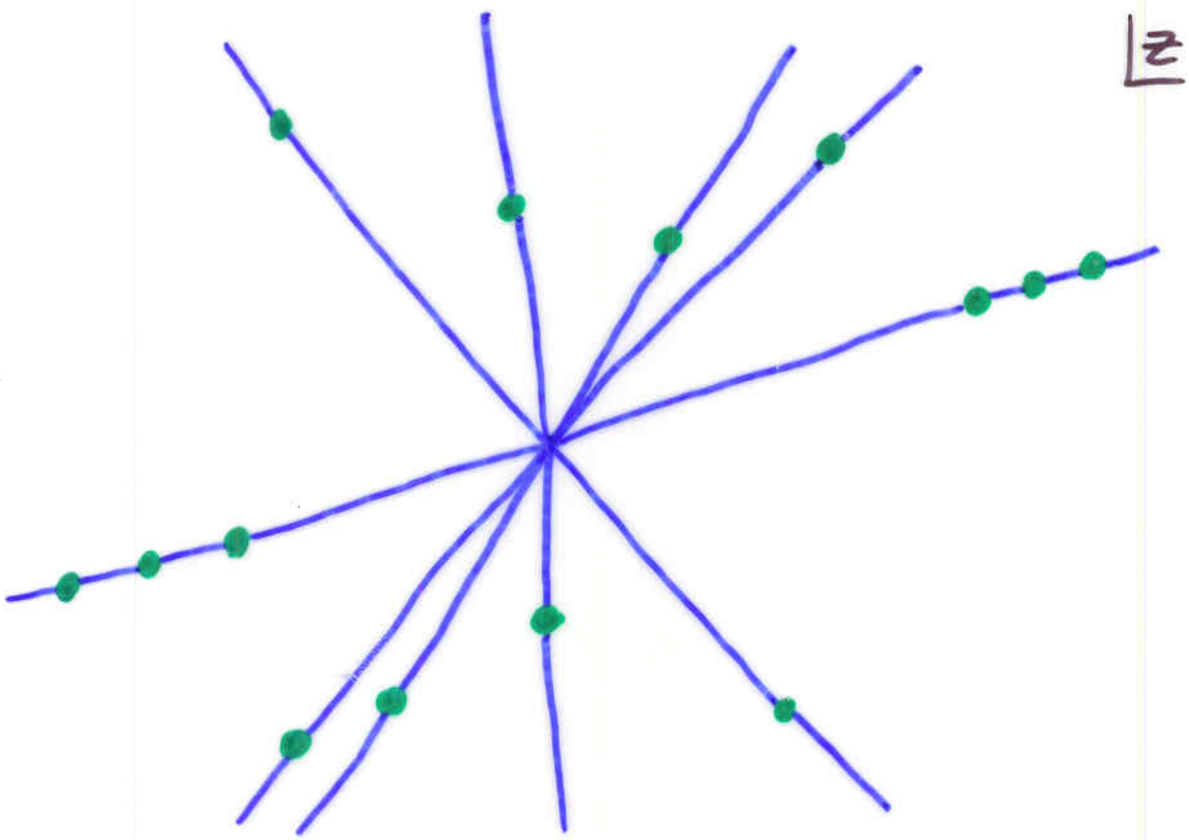
Theorem (Balser-Jurkat-Lutz '79, PB '02)

$$\left\{ \left(\frac{A_0}{z^2} + \frac{B}{z} + H \right) dz \;\middle|\; \begin{array}{l} B \in \mathfrak{g} \\ H : \Delta \to \mathfrak{g} \text{ holom.} \end{array} \right\} \Big/ \mathcal{G}_1$$

$$\cong \quad U_+ \times U_- \times \mathfrak{h} \ni (\underbrace{S_+, S_-}_{\text{Stokes multipliers}}, \underbrace{\Lambda}_{\pi_{\mathfrak{h}}(B)})$$

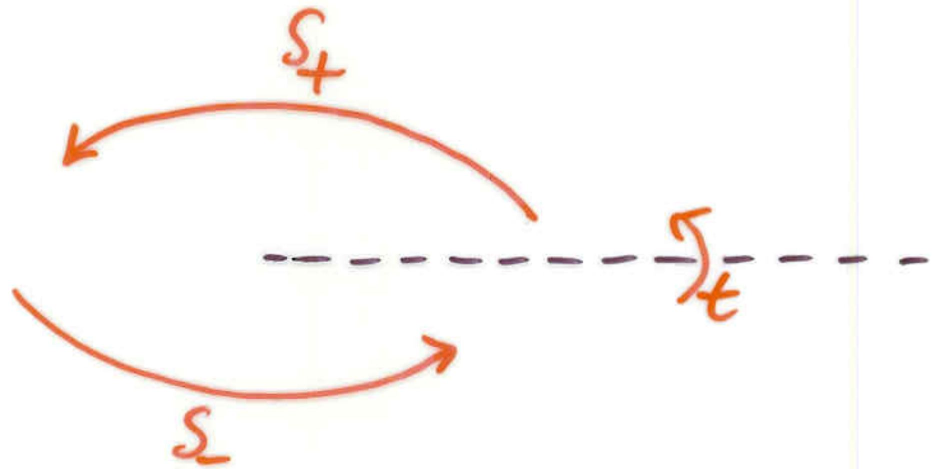
- still have action of $H \subset \mathcal{G}$ "compatible framing at 0"
- to get $\mathcal{C}$: include arbitrary framing at point $1 \in \partial\Delta$ & quotient by H fixing value of Λ $(t = e^{2\pi i \Lambda})$
- have more intrinsic viewpoint (isom. here depends on choice of sector at 0 ⇒ choice of +ve roots)

$$\left.\begin{array}{r}\text{Roots } R \subset \mathfrak{t}^* \\ A_0 \in \mathfrak{t}_{reg}\end{array}\right\} \quad \langle R, A_0 \rangle \subset \mathbb{C}^*$$



Preferred horizontal section on each sector

Φ : sector $\longrightarrow G$ (asymptotics / k-summation)



s_+
t
s_-

$\rightsquigarrow$ $U_+ \times U_- \times H$ inherits complex Poisson str.

Theorem This Poisson manifold is isomorphic to the standard Poisson Lie group G^* (dual to G).

$$G^* = \left\{ (b_+, b_-, \Lambda) \in B_+ \times B_- \times H \;\middle|\; \pi_H(b_+) = \pi_H(b_-^{-1}) = e^{\pi i \Lambda} \right\}$$

- symplectic leaves got by fixing conjugacy class of
$$b_-^{-1} b_+ = S_- S_+ t \in G$$

- (well) approximated by setting $\Theta = 0$, $B \in \mathfrak{g} \cong \mathfrak{g}^*$

map
$$\nu_{A_0} : \mathfrak{g}^* \longrightarrow G^*$$
$$B \longmapsto \left[\frac{A_0}{z^2} + \frac{B}{z} \right]$$

is Poisson for any $A_0 \in \mathfrak{h}_{reg}$

"dual exponential map"

More hyperkähler spaces (examples)

- Typically $\begin{array}{ccccc} \mathcal{O} & \overset{\nu}{\hookrightarrow} & \mathcal{L} & \subset & G^* \\ \cap & & & & \\ \mathfrak{g}^* & & & & \end{array}$

Forget framings: $M \cong \mathcal{L} /\!\!/_{\!\!1} H$ (hk)

(partial compactification of $\mathcal{O} /\!\!/_{\!\!0} H$)

- 2 order 2 poles on $\mathbb{P}^1$:-

with compatible framings $\tilde{M} \cong \Gamma$ "double symplectic groupoid" - Lu-Weinstein

& have $T^*G \hookrightarrow \Gamma$

Forget framings: $M \cong {}_H\!\!/\!\!/ \Gamma /\!\!/ _H$ (hk)

(partial compactification of ${}_H\!\!/\!\!/ T^*G /\!\!/ _H$)

Fourier-Laplace transform

$(GL_n,\ \mathbb{P}^1,\ 1 \text{ order } 2 \text{ pole} + 1 \text{ order } 1 \text{ pole})$

(Balser-Jurkat-Lutz '81)

$A_0 = \mathrm{diag}(a_1, \dots, a_n)$

$$\begin{tikzcd}[column sep=small]
 & B \in \mathfrak{g} \cong \mathfrak{g}^* \arrow[dl, "A_i = E_i B"'] \arrow[dr] & \\
\displaystyle\sum_1^n \frac{A_i}{z-a_i}\,dz \arrow[rr, leftrightarrow, "\text{Fourier-Laplace}"'] \arrow[d, squiggly, "\text{Riemann-Hilbert}"] & & \left(\frac{A_0}{z^2} + \frac{B}{z}\right)dz \arrow[d, squiggly, "\text{Stokes}"'] \\
\begin{array}{c} r_1, \dots, r_n \\ r_i = 1 + e_i \otimes \alpha_i \end{array} \arrow[rr, leftrightarrow, "\text{Killing-Coxeter}", "(\alpha_i(e_j)) = S_+ t - S_-^{-1}"'] \arrow[dr, "r_n \cdots r_1"'] & & S_+, S_-, t \in U_+ \times U_- \times H \arrow[dl, "S_- S_+ t"] \\
 & G &
\end{tikzcd}$$

- Get isomorphisms of moduli spaces
- Led to search for independent general construction
- "Fourier-Laplace is an isometry" now proved by S. Szabo '05
- original interest was via Dubrovin's work on semi-simple Frobenius manifolds (~ '95)

Deligne - Simpson problem

$(GL_n, \mathbb{P}^1, \text{simple poles})$

"When are the spaces non-empty?"

1. Additive version: fix orbits $\mathcal{O}_1, \dots, \mathcal{O}_m \subset \mathfrak{g}$

$\exists ?\ A_1, \dots, A_m$ s.t. $A_i \in \mathcal{O}_i$, $\sum A_i = 0$, irreducible

$$\mathcal{M}^* = \mathcal{O}_1 \times \dots \times \mathcal{O}_m /\!\!/ G \ni \left[\sum \frac{A_i}{z-a_i} dz \right]$$

2. Multiplicative version: fix classes $\mathcal{C}_1, \dots, \mathcal{C}_m \subset G$

$\exists ?\ C_1, \dots, C_m$ s.t $C_i \in \mathcal{C}_i$, $\prod C_i = 1$, irred.

$$M = \mathcal{C}_1 \circledast \dots \circledast \mathcal{C}_m /\!\!/ G$$

Lots of progress made by Crawley-Boevey (solved 1, announced 2) using link to quiver varieties (star-shaped quivers)

(Riemann-Hilbert maps 1 into 2, if non-resonant)

More general problem

$\left(GL_n,\ \mathbb{P}^1,\ 1 \text{ pole of order } 2 + m \text{ simple poles}\right)$

now leading coef. $A_0 \in \mathfrak{t}$, not necessarily $\mathfrak{t}_{reg}$

Let $H(\underset{\sim}{n}) = \mathrm{Stab}(A_0) \cong \prod GL_{n_i} \subset GL_n$

$\underset{\sim}{n} = (n_1, \dots, n_k)$, $\sum n_i = n$ (eigenvalue mults)

$\mathfrak{h}(\underset{\sim}{n}) = \mathrm{Lie}(H(\underset{\sim}{n})) = \ker(\mathrm{ad}_{A_0})$

1. Fix adjoint orbits $\mathcal{O}_1, \dots, \mathcal{O}_m \subset \mathfrak{g}$ & $\check{\mathcal{O}} \subset \mathfrak{h}(\underset{\sim}{n})$

$(\check{\mathcal{O}} \cong \check{\mathcal{O}}_1 \times \dots \times \check{\mathcal{O}}_k,\ \ \check{\mathcal{O}}_i \subset \mathfrak{gl}_{n_i})$

$\exists ?\ A_1, \dots, A_m$ s.t $A_i \in \mathcal{O}_i$, $\pi_{\mathfrak{h}(\underset{\sim}{n})}\left(\sum A_i\right) \in \check{\mathcal{O}}$, irred.

$$\mathcal{M}^* = \mathcal{O}_1 \times \dots \times \mathcal{O}_m \mathbin{/\!\!/}_{\check{\mathcal{O}}} H(\underset{\sim}{n})$$

Key point: These are also quiver varieties

(A_0 scalar ($k=1$), or (by Fourier-Laplace) $m=1$ get original problem)

2 Choose conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_m \subset G$, $\check{\mathcal{C}} \subset H(\underset{\sim}{n})$

$(\check{\mathcal{C}} \cong \check{\mathcal{C}}_1 \times \dots \times \check{\mathcal{C}}_k, \quad \check{\mathcal{C}}_i \subset GL_{n_i})$

$\exists ?\ C_1, \dots, C_m$ s.t $C_i \in \mathcal{C}_i$, "$\pi_{H(\underset{\sim}{n})}(\overrightarrow{\prod} C_i) \in \check{\mathcal{C}}$"

+ irreducible

explicitly: need ① $\overrightarrow{\prod} C_i \in \underset{\sim}{n}$-big cell

$U_+(\underset{\sim}{n}) H(\underset{\sim}{n}) U_-(\underset{\sim}{n})$

② $H(\underset{\sim}{n})$ component is in $\check{\mathcal{C}}$

(affine) Quiver Varieties (... Nakajima)

Quiver = directed finite graph

$$\left.\begin{array}{l} E = \text{edges} \\ I = \text{vertices} \end{array}\right\} \quad E \overset{h}{\underset{t}{\rightrightarrows}} I$$

head

tail

Choose cx vector space V_i $(\forall i \in I)$ $\quad d_i = \dim V_i \Rightarrow \underset{\sim}{d}$

Maps in both directions: $\quad \mathbb{V} = T^* \bigoplus_{e \in E} \mathrm{Hom}\left(V_{t(e)}, V_{h(e)}\right) \ni (g, g')$

$$G = G(\mathbb{V}) = \prod_{i \in I} GL(V_i) \circlearrowleft \mathbb{V}$$

Moment map μ, with components $\mu_i : \mathbb{V} \longrightarrow \mathrm{End}(V_i)$;

$$\mu_i(g, g') = \sum_{e \mid h(e) = i} g_e g'_e - \sum_{e \mid t(e) = i} g'_e g_e$$

Choose scalar at each node $\underset{\sim}{\lambda} = \{\lambda_i \in \mathbb{C} \mid i \in I\}$

$$QV(\underset{\sim}{\lambda}, \underset{\sim}{d}) = \mathbb{V} /\!\!/_{\underset{\sim}{\lambda}} G(\mathbb{V})$$

e.g. • ALE spaces, quiver = affine Dynkin diagram (Kronheimer)

• Closures of adjoint orbits $\mathcal{O} \subset \mathfrak{gl}_n$ (Kraft-Procesi, ...)

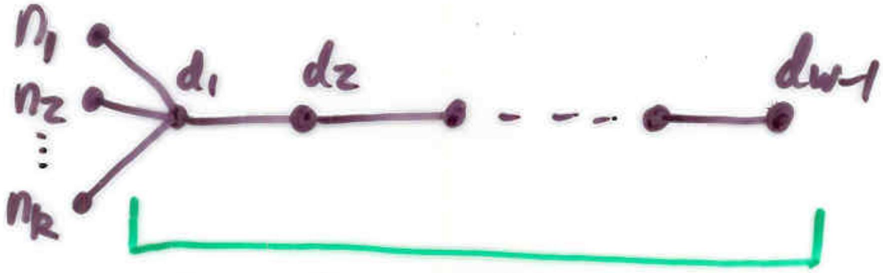
$d_i = \text{rank}\,(A - x_1)\cdots(A - x_i)$ where

$A \in \mathcal{O}$, $x_1, \ldots, x_w$ roots of min poly of A

(params $\underset{\sim}{\lambda}$ ~ evals)

Equivalently take graph:

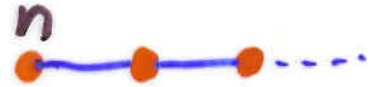
or:



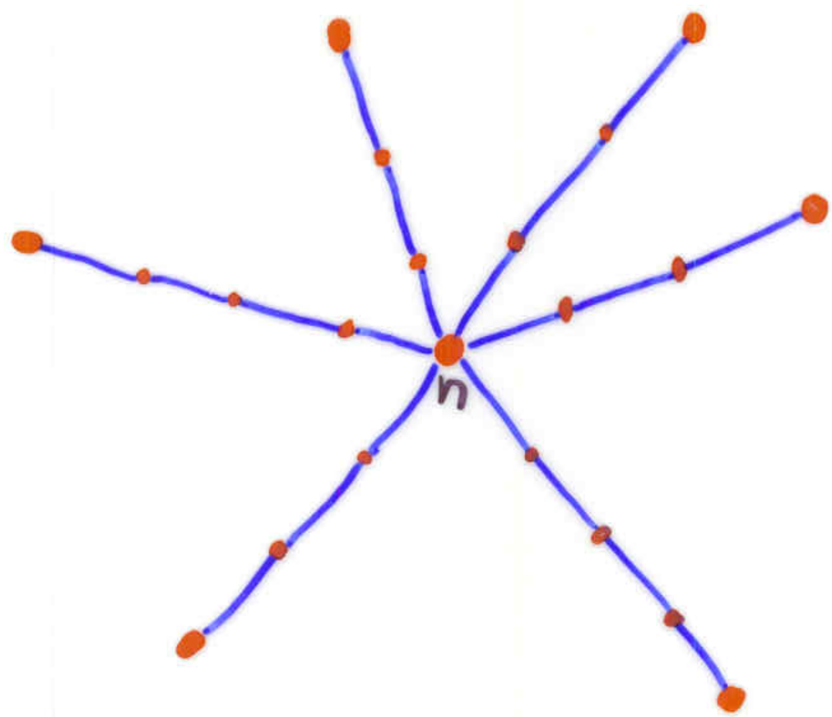
$\sum n_i = n$

and just symplectic quotient by the groups at these nodes

Star shaped quivers

$\mathcal{O}_1, \dots, \mathcal{O}_m \Rightarrow m$ legs 

with residual GL_n action.



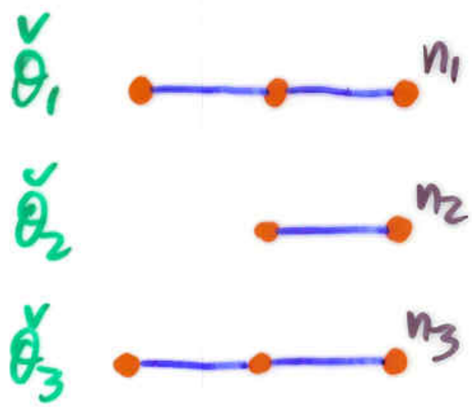
$$QV \cong \mathcal{O}_1 \times \dots \times \mathcal{O}_m /\!\!/ GL_n$$ (if orbits closed)

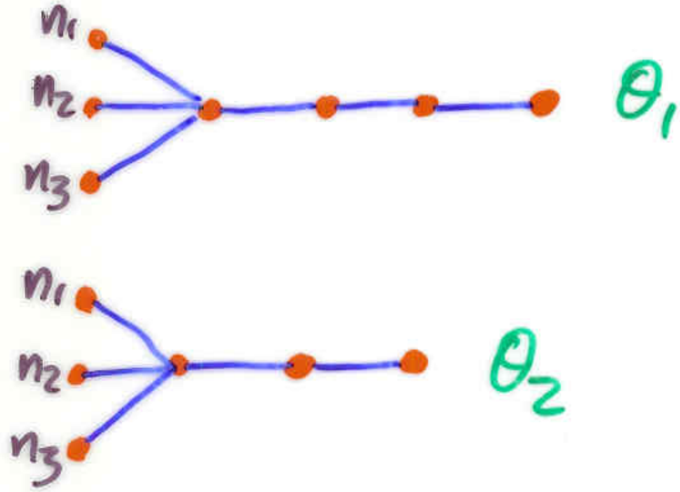
Supernova shaped quivers

want $QV = \mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!\!/_{\check{\mathcal{O}}} H(\underline{n})$

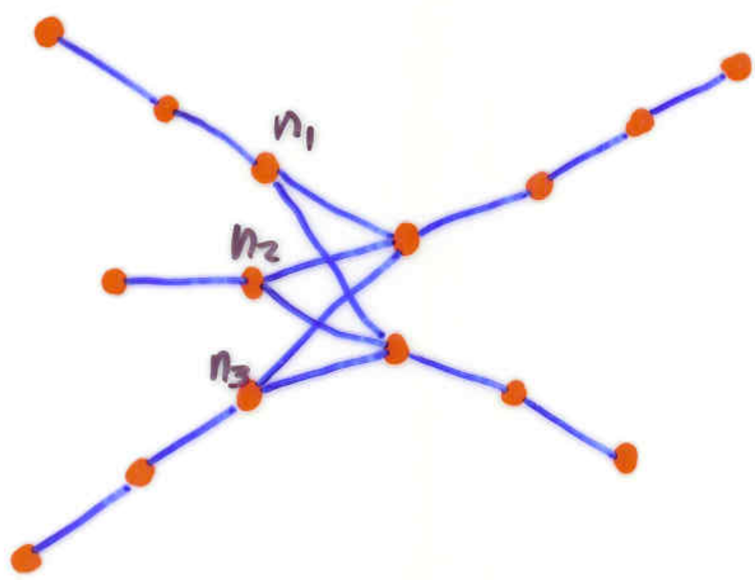
$$\begin{cases} \sum_1^k n_i = n \\ \check{\mathcal{O}} = \prod_1^k \check{\mathcal{O}}_i \\ \check{\mathcal{O}}_i \subset \mathfrak{gl}_{n_i} \end{cases}$$

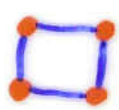
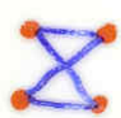
$m=2,\ k=3$:-



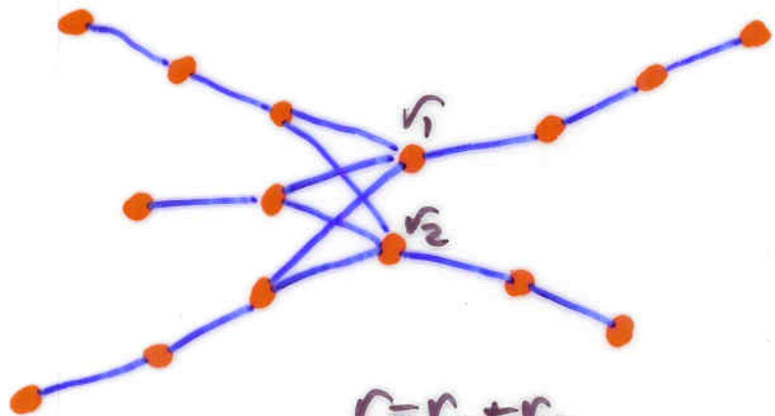


-residual $H(\underline{n})$ action



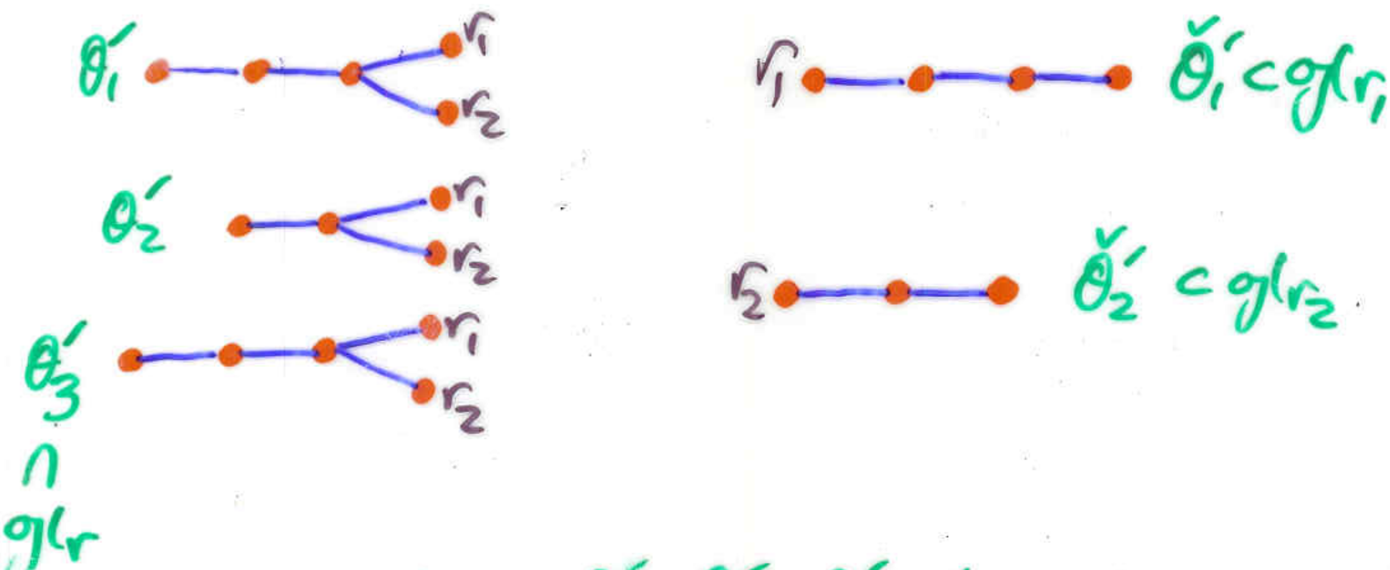
(e.g. $\hat{A}_3$  = )

Dual viewpoint (cf. Harnad)



$r = r_1 + r_2$

$\Uparrow$



so $QV \cong \theta_1' \times \theta_2' \times \theta_3' /\!\!/_{\check{\theta}'} H(\underset{\sim}{r})$ as well

Claim This duality is the Fourier-Laplace transform

(Generalisation of Gelfand-MacPherson duality)